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Infinite Hopf family of elliptic algebras and bosonization

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Abstract. Elliptic current algebras $\mathcal{E}_{q,p}(\hat{g})$ for arbitrary simply laced finite-dimensional Lie algebra g are defined and their co-algebraic structures are studied. Putting the algebras $\mathcal{E}_{q,p}(\hat{g})$ with different deformation parameters together, we establish a structure of an infinite Hopf family of algebras with a Drinfeld-like co-multiplication. The level 1 bosonic realization for the algebra $\mathcal{E}_{q,p}(\hat{g})$ is also established.

1. Introduction

In this paper we continue our recent study on an infinite Hopf family of algebras and obtain new examples of such families—the infinite Hopf family of elliptic algebras.

The concept of an infinite Hopf family of algebras was first introduced in our earlier paper [1] in which the algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ were proposed and their co-algebraic structures were specified. In contrast to the standard Hopf structures for quantum affine algebras and Yangian doubles, algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$, including their most degenerated case $\mathcal{A}_{\hbar,\eta}(\hat{sl_2})$ [2], are not evidently co-closed and their co-algebraic structures are formulated in terms of some generalized Hopf structure, examples are the Hopf family of algebras of [2] and the infinite Hopf family of algebras of our paper [1].

The algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ which appeared in [1] are very unusual. For $g = sl_2$, such algebra was proposed as the scaling limit of the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl_2})$ and thus inherits *two* deformation parameters $\hbar$ and η. The first parameter $\hbar$ can be viewed as a 'quantization parameter', because in the limit $\hbar \to 0$, the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ becomes a classical algebra. The second parameter η should be viewed as a 'family deformation parameter' because the set of algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ with different η form the first known non-trivial example of the infinite Hopf family of algebras and while $\eta \to 0$ the family structure becomes trivial. Another unusual feature of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ is that, under current realization, the generating currents corresponding to positive and negative roots are deformed *differently*. Despite their unusual mathematical features, $\mathcal{A}_{\hbar,\eta}(\hat{g})$ are believed to have important applications in integrable quantum field theories such as sine–Gordon and affine-Toda field theories [1, 2]. Moreover, the study of such algebras can provide a better understanding of the $((\hbar, \xi)$-) deformed Virasoro and W algebras which have recently been under active study.

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In this paper we are motivated to study both the elliptic generalizations of the algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ (or the pre-scaling algebras) [1] and their associated infinite Hopf family of algebras.

The search for elliptic quantum algebras has lasted for some years. Various elliptic deformed algebras have emerged in several different contexts, among them are the Sklyanin algebras of type sl_N, the algebra $\mathcal{A}_{q,p}(\hat{sl}_2)$ [3, 4] and its generalization to the sl_N case, $\mathcal{A}_{q,p}(\hat{sl}_N)$, which forms the class of so-called vertex-type elliptic algebras and the 'elliptic quantum groups' of Felder *et al* [5–7] and the dynamical twisted algebra of Hou *et al* [8] which form the class of so-called face-type elliptic algebras. The mentioned elliptic algebras are all realized through (vertex- and face-type) Yang–Baxter relations. The difference between Sklyanin algebra and $\mathcal{A}_{q,p}(\hat{sl}_N)$, as well as that between Felder *et al* algebra and Hou *et al* ones lie in that, the modulus for the elliptic entries of R-matrices are the same for the former and are different for the latter algebras. Other examples of elliptic algebras are those for the deformed screening currents for quantum deformed W algebras (defined for any simply-laced underlying Lie algebra g) of the first two by the present authors [9] and Konno's algebra $U_{q,p}(\hat{sl}_2)$ [10].

We note that the classification for elliptic deformed algebras seems far from complete. For example, the last two types of algebras are realized as current algebras only and their possible Yang–Baxter-type realization are still unknown. Moreover, though the co-structures, or more explicitly, the quasi-Hopf structures, for the vertex- and face-type algebras realized through Yang–Baxter-type relations have recently been clarified due to the work of Fronsdal and Jombo *et al*, similar structures in the current algebras of [9] are still unknown.

In this paper, we present a new type of elliptic current algebras which we denote as $\mathcal{E}_{q,p}(\hat{g})$ (where g can be any classical simply-laced Lie algebra) and study the associated infinite Hopf family of algebraic structures. It turns out that $\mathcal{E}_{q,p}(\hat{g})$ are quite similar to the algebras of modified screening currents for the quantum (q, p)-deformed W-algebras mentioned earlier in level 1 bosonic representations. The only difference lies in that, for the algebras $\mathcal{E}_{q,p}(\hat{g})$ at level 1, the deformation parameter q is the inverse of the one in the algebras defined in [9] (whilst the parameter $\tilde{q}$ is kept unchanged) and we assume here that $|q| < 1$, which corresponds to $|q| > 1$ in [9] (the algebras in [9], however, were defined only for $|q| < 1$ implicitly). This slight difference prevented us from defining the somewhat well expected structure of the infinite Hopf family of algebras in [9].

The organization of this paper is as follows. In section 2, we give a definition for the current algebra $\mathcal{E}_{q,p}(\hat{g})$. Section 3 is devoted to the study of the structure of the associated infinite Hopf family of algebras. In section 4 we give the bosonic representation for the current algebras $\mathcal{E}_{q,p}(\hat{g})$ at level 1. The final section—section 5—is for concluding remarks.

2. The elliptic current algebra $\mathcal{E}_{q,p}(\hat{g})$

We first give the definition for the elliptic current algebras $\mathcal{E}_{q,p}(\hat{g})$.

Definition 2.1. *Let g be any finite-dimensional simply-laced Lie algebra with Cartan matrix (A_{ij}). The elliptic current algebra $\mathcal{E}_{q,p}(\hat{g})$ is the associative algebra generated by the currents $E_i(z)$, $F_i(z)$, $H_i^{\pm}(z)$ with $i = 1, 2, \ldots, rank(g)$, central element c and unit element 1 with the following relations*

$$H_i^{\pm}(z)H_j^{\pm}(w) = \frac{\theta_q((z/w)p^{A_{ij}/2})\theta_{\tilde{q}}((z/w)p^{-A_{ij}/2})}{\theta_q((z/w)p^{-A_{ij}/2})\theta_{\tilde{q}}((z/w)p^{A_{ij}/2})}H_j^{\pm}(w)H_i^{\pm}(z) \tag{1}$$

$$H_i^{+}(z)H_j^{-}(w) = \frac{\theta_q((z/w)p^{(A_{ij}+c)/2})\theta_{\tilde{q}}((z/w)p^{-(A_{ij}+c)/2})}{\theta_q((z/w)p^{-(A_{ij}-c)/2})\theta_{\tilde{q}}((z/w)p^{(A_{ij}-c)/2})}H_j^{-}(w)H_i^{+}(z) \tag{2}$$

$$H_i^+(z)E_j(w) = (-1)^{A_{ij}}(p^{-A_{ij}/2})\frac{\theta_q((z/w)p^{A_{ij}/2}p^{c/4})}{\theta_q((z/w)p^{-A_{ij}/2}p^{c/4})}E_j(w)H_i^+(z) \tag{3}$$

$$H_i^-(z)E_j(w) = (-1)^{A_{ij}}(p^{-A_{ij}/2})\frac{\theta_q((z/w)p^{A_{ij}/2}p^{-c/4})}{\theta_q((z/w)p^{-A_{ij}/2}p^{-c/4})}E_j(w)H_i^-(z) \tag{4}$$

$$H_i^+(z)F_j(w) = (-1)^{A_{ij}}(p^{A_{ij}/2})\frac{\theta_{\tilde{q}}((z/w)p^{-A_{ij}/2}p^{-c/4})}{\theta_{\tilde{q}}((z/w)p^{A_{ij}/2}p^{-c/4})}F_j(w)H_i^+(z) \tag{5}$$

$$H_i^-(z)F_j(w) = (-1)^{A_{ij}}(p^{A_{ij}/2})\frac{\theta_{\tilde{q}}((z/w)p^{-A_{ij}/2}p^{c/4})}{\theta_{\tilde{q}}((z/w)p^{A_{ij}/2}p^{c/4})}F_j(w)H_i^-(z) \tag{6}$$

$$E_i(z)E_j(w) = (-1)^{A_{ij}}(p^{-A_{ij}/2})\frac{\theta_q((z/w)p^{A_{ij}/2})}{\theta_q((z/w)p^{-A_{ij}/2})}E_j(w)E_i(z) \tag{7}$$

$$F_i(z)F_j(w) = (-1)^{A_{ij}}(p^{A_{ij}/2})\frac{\theta_{\tilde{q}}((z/w)p^{-A_{ij}/2})}{\theta_{\tilde{q}}((z/w)p^{A_{ij}/2})}F_j(w)F_i(z) \tag{8}$$

$$(E_i(z), F_j(w)) = \frac{\delta_{ij}}{(p-1)zw}\left(\delta\left(\frac{z}{w}p^{c/2}\right)H_i^+(wp^{c/4}) - \delta\left(\frac{w}{z}p^{-c/2}\right)H_i^-(zp^{-c/4})\right) \tag{9}$$

$$E_i(z_1)E_i(z_2)E_j(w) - f_{ij}^{(q)}(z_1/w, z_2/w)E_i(z_1)E_j(w)E_i(z_2) + E_j(w)E_i(z_1)E_i(z_2) \\ + (\textit{replacement})\ z_1 \leftrightarrow z_2) = 0 \qquad A_{ij} = -1 \tag{10}$$

$$F_i(z_1)F_i(z_2)F_j(w) - f_{ij}^{(\tilde{q})}(z_1/w, z_2/w)F_i(z_1)F_j(w)F_i(z_2) + F_j(w)F_i(z_1)F_i(z_2) \\ + (\textit{replacement})\ z_1 \leftrightarrow z_2) = 0 \qquad A_{ij} = -1 \tag{11}$$

where

$$f_{ij}^{(a)}(z_1/w, z_2/w) = \frac{(\psi_{ii}^{(a)}(z_2/z_1)+1)(\psi_{ij}^{(a)}(w/z_1)\psi_{ij}^{(a)}(w/z_2)+1)}{\psi_{ij}^{(a)}(w/z_2)+\psi_{ii}^{(a)}(z_2/z_1)\psi_{ij}^{(a)}(w/z_1)} \qquad a = q, \tilde{q}$$

$$\psi_{ij}^{(q)}(x) = (-1)^{A_{ij}}p^{-A_{ij}/2}\frac{\theta_q(x^{-1}p^{A_{ij}/2})}{\theta_q(x^{-1}p^{-A_{ij}/2})}$$

$$\psi_{ij}^{(\tilde{q})}(x) = (-1)^{A_{ij}}p^{A_{ij}/2}\frac{\theta_{\tilde{q}}(x^{-1}p^{-A_{ij}/2})}{\theta_{\tilde{q}}(x^{-1}p^{A_{ij}/2})}$$

q and p are a pair of deformation parameters with norms $|q| < 1$ and $|p| < 1$, z, w are spectral parameters, $\tilde{q}$ and q are connected by the relation

$$\tilde{q}/q = p^c$$

and $\theta_q(z)$ is the standard elliptic function given by

$$\theta_q(z) = (z|q)_\infty(qz^{-1}|q)_\infty(q|q)_\infty$$

$$(z|q_1, \ldots, q_m)_\infty = \prod_{i_1,i_2,\ldots,i_m=0}^{\infty}(1 - zq_1^{i_1}q_2^{i_2}\ldots q_m^{i_m}).$$

Quite analogous to the case of $\mathcal{A}_{\hbar,\eta}(\hat{g})$, the elliptic current algebra given previously enjoys the following features:

- it has *two* deformation parameters p, q and ‘positive’ and ‘negative’ currents $E(z)$ and $F(z)$ are deformed differently (each corresponds to one of two parameters q and $\tilde{q}$, respectively);
- the currents $H_i^{\pm}(z)$ do not commute in contrast to the q-affine and Yangian cases.

These features are also shared by the algebras $\mathcal{A}_{q,p}(\hat{sl}_N)$, $\mathcal{A}_{q,p,\hat{\pi}}(\hat{sl}_2)$ and $U_{q,p}(\hat{sl}_2)$.

The second feature has a rather significant consequence. If one considers the subalgebras generated by the currents $H^+(z), E(z)$ or $H^-(z), F(z)$, it would turn out that they do not form nilpotent or even solvable subalgebras. However, in q-affine and Yangian cases similar subalgebras are indeed solvable and, with the aid of a properly defined Manin pairing, they give rise to the structure of quantum doubles. The non-solvability of such subalgebras in our case might imply that the algebra $\mathcal{E}_{q,p}(\hat{g})$ under consideration does not have a simple quantum double structure.

In order to show the deeper relationship between our algebra and $\mathcal{A}_{\hbar,\eta}(\hat{g})$, we give the following proposition which shows that the algebra $\mathcal{E}_{q,p}(\hat{g})$ is an elliptic extension of $\mathcal{A}_{\hbar,\eta}(\hat{g})$.

Proposition 2.2. *In the scaling limit*

$$p = \mathrm{e}^{\epsilon\hbar} \qquad q = \mathrm{e}^{\epsilon/\eta} \qquad z = \mathrm{e}^{\mathrm{i}\epsilon u}$$
$$\epsilon \to 0$$

the algebra $\mathcal{E}_{q,p}(\hat{g})$ tends to the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ defined in [1] up to terms linear in ϵ.

We remark that for the case $g = sl_2$, both $\mathcal{A}_{q,p}(\hat{sl_2})$ and $U_{q,p}(\hat{sl_2})$ would yield $\mathcal{A}_{\hbar,\eta}(\hat{sl_2})$ in the scaling limit. Therefore, our algebra $\mathcal{E}_{q,p}(\hat{g})$ has the same scaling limit as those two algebras for the special underlying Lie algebra $g = sl_2$. However, for general simply-laced g, $\mathcal{E}_{q,p}(\hat{g})$ is the only known algebra which tends to $\mathcal{A}_{\hbar,\eta}(\hat{g})$ in the scaling limit. Actually, the generalization of $\mathcal{A}_{q,p}(\hat{sl_2})$ to the case of D, E series of Lie algebras are not known to exist. Likewise, the generalization of $U_{q,p}(\hat{sl_2})$ to any other g is also not known to exist. (We noticed the similarity between our algebra at $g = sl_2$ and $U_{q,p}(\hat{sl_2})$. It is possible that these two algebras are isomorphic, however, we do not make this claim because it has yet to be proven†.)

Another remark is in order here. The algebra $\mathcal{E}_{q,p}(\hat{g})$, as well as $\mathcal{A}_{\hbar,\eta}(\hat{g})$ defined in [1], should be regarded as *current* algebras only since we do not know the corresponding Yang–Baxter-type realizations. Actually, given a Yang–Baxter-type relation one can define an associative algebra which is a certain deformation of the universal enveloping algebra of some underlying Lie algebra and, due to the well known Ding–Frenkel homomorphism, one can find a corresponding current realization which is an important usage for the construction of infinite-dimensional representations. However, the inverse to Ding–Frenkel homomorphism is some Riemann problem which often does not possess a unique solution [11]. Therefore, given the definition of a current algebra such as $\mathcal{E}_{q,p}(\hat{g})$, one actually cannot associate a unique Yang–Baxter-type relation without putting in extra constraints. It seems quite possible that both the vertex-type and face-type elliptic algebras can be obtained from the same current algebra $\mathcal{E}_{q,p}(\hat{g})$ by introducing different sets of constraints which lead to different solutions to the Riemann problem. We hope to consider this problem in later studies.

3. The structure of the infinite Hopf family of algebras for $\mathcal{E}_{q,p}(\hat{g})$

The algebra $\mathcal{E}_{q,p}(\hat{g})$ defined in the last section is in fact the representative of an infinite Hopf family of elliptic algebras which we now specify.

Let $\{\mathcal{A}_n, n \in Z\}$ be a family of associative algebras over C with unity. Let $\{v_i^{(n)}, i = 1, \ldots, \dim(\mathcal{A}_n)\}$ be a basis of $\mathcal{A}_n$. The maps

$$\tau_n^{\pm} : \mathcal{A}_n \to \mathcal{A}_{n\pm1}$$
$$v_i^{(n)} \mapsto v_i^{(n\pm1)}$$

† To compare with [10], we should bear in mind that the following change of notation should be made: $q \to p$, $p \to q^2$ and $c \to -c$.

are morphisms from $\mathcal{A}_n$ to $\mathcal{A}_{n\pm1}$. For any two integers n, m with $n < m$, we can specify a pair of morphisms

$$\mathrm{Mor}(\mathcal{A}_m, \mathcal{A}_n) \ni \tau^{(m,n)} \equiv \tau^+_{m-1} \dots \tau^+_{n+1}\tau^+_n : \mathcal{A}_n \to \mathcal{A}_m$$
$$\mathrm{Mor}(\mathcal{A}_n, \mathcal{A}_m) \ni \tau^{(n,m)} \equiv \tau^-_{n+1} \dots \tau^-_{m-1}\tau^-_m : \mathcal{A}_m \to \mathcal{A}_n \qquad (12)$$

with $\tau^{(m,n)}\tau^{(n,m)} = id_m, \tau^{(n,m)}\tau^{(m,n)} = id_n$. Clearly the morphisms $\tau^{(m,n)}, n, m \in Z$ satisfy the associativity condition $\tau^{(m,p)}\tau^{(p,n)} = \tau^{(m,n)}$ and thus make the family of algebras $\{\mathcal{A}_n, n \in Z\}$ into a category.

Definition 3.1. *The category of algebras* $\{\mathcal{A}_n, \{\tau^{(n,m)}\}, n, m \in Z\}$ *is called an infinite Hopf family of algebras if on each object* $\mathcal{A}_n$ *of the category one can define the morphisms* $\Delta^\pm_n : \mathcal{A}_n \to \mathcal{A}_n \otimes \mathcal{A}_{n\pm1}$, $\epsilon_n : \mathcal{A}_n \to C$ *and antimorphisms* $S^\pm_n : \mathcal{A}_n \to \mathcal{A}_{n\pm1}$ *such that the following axioms hold*

- $(\epsilon_n \otimes id_{n+1}) \circ \Delta^+_n = \tau^+_n \qquad (id_{n-1} \otimes \epsilon_n) \circ \Delta^-_n = \tau^-_n$ (a1)
- $m_{n+1} \circ (S^+_n \otimes id_{n+1}) \circ \Delta^+_n = \epsilon_{n+1} \circ \tau^+_n \qquad m_{n-1} \circ (id_{n-1} \otimes S^-_n) \circ \Delta^-_n = \epsilon_{n-1} \circ \tau^-_n$ (a2)
- $(\Delta^-_n \otimes id_{n+1}) \circ \Delta^+_n = (id_{n-1} \otimes \Delta^+_n) \circ \Delta^-_n$ (a3)

in which m_n *is the algebra multiplication for* $\mathcal{A}_n$.

Remark 3.2. We remark here that the presentation of an infinite Hopf family of algebras is slightly different from that of [1] in the trigonometric case. However, the statement that the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ is representative of an infinite Hopf family of trigonometric algebras still holds true under the present definition of the infinite Hopf family of algebras.

Let $\mathcal{A}$ be an associative algebra over C with unity. A trivial example of an infinite Hopf family of algebras is given by the category of algebras $\{\mathcal{A}_n \equiv \mathcal{A}, \{\tau^{(n,m)} \equiv id_{\mathcal{A}}\}, n, m \in Z\}$ with $\Delta^\pm_n$, ϵ_n and $S^\pm_n$ identified as the standard Hopf algebra structures over $\mathcal{A}$. This trivial example shows that the infinite Hopf family of algebras can be regarded as a deformation of the standard Hopf algebra structure. The maps $\Delta^\pm_n$, ϵ_n and $S^\pm_n$ in the infinite Hopf family of algebras are called co-multiplications, co-units and antipodes by this analogy.

Now, let us consider the infinite Hopf family of algebras structure of $\mathcal{E}_{q,p}(\hat{g})$. For this purpose we introduce some notation. First, we denote the algebra $\mathcal{E}_{q,p}(\hat{g})$ by $\mathcal{E}_{q,p}(\hat{g})_c$, specifying explicitly the central extension c. We see that this algebra is determined uniquely as a current algebra by the defining equations (1)–(11) provided the following data are fixed: g, q, p, c. In general, given a series of $c_n, n \in Z$, we can define

$$q^{(n+1)}/q^{(n)} = p^{c_n}$$

starting from the data $q^{(1)} = q, c_1 = c$. It is obvious that $\tilde{q} = q^{(2)}$ and hence $\tilde{q}^{(n)} = q^{(n+1)}$. We collect the family of algebras $\{\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}, n \in Z\}$ where $\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}$ is the algebra $\mathcal{E}_{q,p}(\hat{g})_c$ with q replaced by $q^{(n)}$ and c by c_n. The generating currents $H^\pm_i(z)$, $E_i(z)$ and $F_i(z)$ for $\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}$ are denoted as $H^\pm_i(z; q^{(n)})$, $E_i(z; q^{(n)})$ and $F_i(z; q^{(n)})$ etc.

The family of algebras $\{\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}, n \in Z\}$ can be easily turned into a category by introducing the morphisms $\tau^\pm_n$

$$\begin{aligned}
&\tau^\pm_n : \mathcal{E}_{g^{(n)},p}(\hat{g})_{c_n} \to \mathcal{E}_{q^{(n\pm1)},p}(\hat{g})_{c_{n\pm1}} \\
&H^\pm_i(z; q^{(n)}) \mapsto H^\pm_i(z; q^{(n\pm1)}) \\
&E_i(z; q^{(n)}) \mapsto E_i(z; q^{(n\pm1)}) \\
&F_i(z; q^{(n)}) \mapsto F_i(z; q^{(n\pm1)}) \\
&c_n \mapsto c_{n\pm1}
\end{aligned}$$

and defining the compositions $\tau^{(n,m)}$ as in (12).

The following proposition is one of our major results.

Proposition 3.3. *The category of algebras* $\{\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}, \{\tau^{(n,m)}\}, n, m \in Z\}$ *form an (elliptic) infinite Hopf family of algebras with the Hopf family structures given as follows:*

- *the co-multiplications* $\Delta_n^{\pm}$

$$\begin{aligned}
&\Delta_n^+ c_n = c_n + c_{n+1}\\
&\Delta_n^+ H_i^+(z; q^{(n)}) = H_i^+(zp^{c_{n+1}/4}; q^{(n)}) \otimes H_i^+(zp^{-c_n/4}; q^{(n+1)})\\
&\Delta_n^+ H_i^-(z; q^{(n)}) = H_i^-(zp^{-c_{n+1}/4}; q^{(n)}) \otimes H_i^-(zp^{c_n/4}; q^{(n+1)})\\
&\Delta_n^+ E_i(z; q^{(n)}) = E_i(z; q^{(n)}) \otimes 1 + H_i^-(zp^{c_n/4}; q^{(n)}) \otimes E_i(zp^{c_n/2}; q^{(n+1)})\\
&\Delta_n^+ F_i(z; q^{(n)}) = 1 \otimes F_i(z; q^{(n+1)}) + F_i(zp^{c_{n+1}/2}; q^{(n)}) \otimes H_i^+(zp^{c_{n+1}/4}; q^{(n+1)})
\end{aligned}$$

$$\begin{aligned}
&\Delta_n^- c_n = c_{n-1} + c_n\\
&\Delta_n^- H_i^+(z; q^{(n)}) = H_i^+(zp^{c_n/4}; q^{(n-1)}) \otimes H_i^+(zp^{-c_{n-1}/4}; q^{(n)})\\
&\Delta_n^- H_i^-(z; q^{(n)}) = H_i^-(zp^{-c_n/4}; q^{(n-1)}) \otimes H_i^-(zp^{c_{n-1}/4}; q^{(n)})\\
&\Delta_n^- E_i(z; q^{(n)}) = E_i(z; q^{(n-1)}) \otimes 1 + H_i^-(zp^{c_{n-1}/4}; q^{(n-1)}) \otimes E_i(zp^{c_{n-1}/2}; q^{(n)})\\
&\Delta_n^- F_i(z; q^{(n)}) = 1 \otimes F_i(z; q^{(n)}) + F_i(zp^{c_n/2}; q^{(n-1)}) \otimes H_i^+(zp^{c_n/4}; q^{(n)});
\end{aligned}$$

- *the co-units* ϵ_n

$$\begin{aligned}
&\epsilon_n(c_n) = 0\\
&\epsilon_n(1_n) = 1\\
&\epsilon_n(H_i^{\pm}(z; q^{(n)})) = 1\\
&\epsilon_n(E_i(z; q^{(n)})) = 0\\
&\epsilon_n(F_i(z; q^{(n)})) = 0;
\end{aligned}$$

- *the antipodes* $S_n^{\pm}$

$$\begin{aligned}
&S_n^{\pm} c_n = -c_{n\pm 1}\\
&S_n^{\pm} H_i^+(z; q^{(n)}) = [H_i^+(z; q^{(n\pm 1)})]^{-1}\\
&S_n^{\pm} H_i^-(z; q^{(n)}) = [H_i^-(z; q^{(n\pm 1)})]^{-1}\\
&S_n^{\pm} E_i(z; q^{(n)}) = -H_i^-(zp^{-c_{n\pm 1}/4}; q^{(n\pm 1)})^{-1} E_i(zp^{-c_{n\pm 1}/2}; q^{(n\pm 1)})\\
&S_n^{\pm} F_i(z; q^{(n)}) = -F_i(zp^{-c_{n\pm 1}/2}; q^{(n\pm 1)}) H_i^+(zp^{-c_{n\pm 1}/4}; q^{(n\pm 1)})^{-1}.
\end{aligned}$$

The proof for this proposition is by straightforward calculations.

Remark 3.4. The co-multiplications, co-units and antipodes given previously are analogous to the Drinfeld–Hopf structures for q-affine algebras. The difference lies in that, instead of sending elements of the algebra $\mathcal{A}_n$ into the tensor product space of the same algebra, the co-multiplications $\Delta_n^{\pm}$ now send elements of $\mathcal{A}_n$ into the tensor product spaces $\mathcal{A}_n \otimes \mathcal{A}_{n+1}$ and $\mathcal{A}_{n-1} \otimes \mathcal{A}_n$, respectively, of two neighbouring algebras in the family. The shift in the suffices in the notation of target spaces indicate the crucial difference between the non-trivial infinite Hopf family of algebras and trivial ones.

In order to understand the meaning of the unusual shift of suffices mentioned before, we present here another proposition which was first found in [1] for the trigonometric case.

Proposition 3.5. *The co-multiplication* Δ_n^+ *induces an algebra homomorphism*

$$\begin{aligned}
&\rho : \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n+c_{n+1}} \to \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n} \otimes \mathcal{E}_{q^{(n+1)},p}(\hat{g})_{c_{n+1}}\\
&X \mapsto \Delta_n^+ \tilde{X}
\end{aligned}$$

where $X \in \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n+c_{n+1}}$, $\tilde{X} \in \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}$ *and*

$$\tilde{X} = \begin{cases} c_n \\ H_i^{\pm}(z; q^{(n)}) \\ E_i(z; q^{(n)}) \\ F_i(z; q^{(n)}) \end{cases} \qquad \text{if } X = \begin{cases} c_n + c_{n+1} \\ H_i^{\pm}(z; q^{(n)}) \\ E_i(z; q^{(n)}) \\ F_i(z; q^{(n)}). \end{cases}$$

Likewise, the co-multiplication Δ_n^- *induces an algebra homomorphism*

$$\begin{aligned} &\bar{\rho} : \mathcal{E}_{q^{(n-1)},p}(\hat{g})_{c_{n-1}+c_n} \to \mathcal{E}_{q^{(n-1)},p}(\hat{g})_{c_{n-1}} \otimes \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n} \\ &X \mapsto \Delta_n^+ \tilde{X} \end{aligned}$$

where $X \in \mathcal{E}_{q^{(n-1)},p}(\hat{g})_{c_{n-1}+c_n}$, $\tilde{X} \in \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}$ *and*

$$\tilde{X} = \begin{cases} c_n \\ H_i^{\pm}(z; q^{(n)}) \\ E_i(z; q^{(n)}) \\ F_i(z; q^{(n)}) \end{cases} \qquad \text{if } X = \begin{cases} c_{n-1} + c_n \\ H_i^{\pm}(z; q^{(n-1)}) \\ E_i(z; q^{(n-1)}) \\ F_i(z; q^{(n-1)}). \end{cases}$$

Corollary 3.6. *Let m be a positive integer. The iterated co-multiplication* $\Delta_n^{(m)+} = (id_n \otimes id_{n+1} \otimes \cdots \otimes id_{n+m-2} \otimes \Delta_{n+m-1}^+) \circ (id_n \otimes id_{n+1} \otimes \cdots \otimes id_{n+m-3} \otimes \Delta_{n+m-2}^+) \cdots \circ (id_n \otimes \Delta_{n+1}^+) \circ \Delta_n^+$ *induces an algebra homomorphism* $\rho^{(m)}$

$$\rho^{(m)} : \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n+c_{n+1}+\cdots+c_{n+m}} \to \mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n} \otimes \mathcal{E}_{q^{(n+1)},p}(\hat{g})_{c_{n+1}} \otimes \cdots \otimes \mathcal{E}_{q^{(n+m)},p}(\hat{g})_{c_{n+m}}$$

in the spirit of proposition 3.5.

Remark 3.7. We stress here that the maps ρ, $\bar{\rho}$ and $\rho^{(m)}$ are *algebra homomorphisms*, whilst $\tau_n^{\pm}$, $\tau^{(n,m)}$ and $\Delta_n^{\pm}$ etc are only algebra morphisms. The difference between algebra morphisms and algebra homomorphisms lies in the fact that the latter preserves the structure functions whilst the former does not. In fact, the algebra morphisms $\tau_n^{\pm}$, $\tau^{(n,m)}$ and $\Delta_n^{\pm}$ etc have non-trivial actions on structure functions, or, more precisely, on the deformation parameters, for example

$$\tau^{(n,m)} : q^{(m)} \to q^{(n)}$$

etc.

This proposition and its corollary shows that although the algebras $\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}$ are not co-closed, the tensor product representations can still be defined using the co-multiplications $\Delta_n^{\pm}$. In particular, if the $c_n = 1$ representation for $\mathcal{E}_{q^{(n)},p}(\hat{g})_{c_n}$ is available, then the representations at any positive integer c_n are all available, leaving aside the reducibility problems of such representations. This statement is of particular importance when one needs to realize that the algebra $\mathcal{E}_{q,p}(\hat{g})_c$ is non-empty for $c \in Z_+ \backslash \{1\}$.

4. Free boson realization of the algebra $\mathcal{E}_{q,p}(\hat{g})$ at $c = 1$

Having established the infinite Hopf family of algebras structure of the elliptic current algebra $\mathcal{E}_{q,p}(\hat{g})$, we now turn to consider its simplest infinite-dimensional representation, i.e. the free boson realization at $c = 1$.

First, we introduce the Heisenberg algebra $\mathcal{H}_{q,p}(g)$ with generators $a_i[n]$, P_i, Q_i, $i = 1, \ldots, \mathrm{rank}(g)$, $n \in Z\backslash\{0\}$ and generating relations

$$\begin{aligned} &[a_i[n], a_j[m]] = \frac{1}{n} \frac{(1 - q^{-n})(p^{nA_{ij}/2} - p^{-nA_{ij}/2})(1 - (pq)^n)}{1 - p^n} \delta_{n,m} \\ &[P_i, Q_j] = A_{ij} \end{aligned}$$

where A_{ij} is the Cartan matrix for the Lie algebra g and all the commutators vanish. Let

$$s_i^+[n] = \frac{a_i[n]}{q^n - 1} \qquad s_i^-[n] = -\frac{a_i[n]}{(pq)^{-n} - 1}$$

and define the (deformed) free boson fields

$$\varphi_i(z) = \sum_{n\neq 0} s_i^+[n] z^{-n} \qquad \psi_i(z) = \sum_{n\neq 0} s_i^-[n] z^{-n}.$$

Proposition 4.1. *The following bosonic expressions give a level $c = 1$ realization for the algebra $\mathcal{E}_{q,p}(\hat{g})$ on the Fock space of the Heisenberg algebra $\mathcal{H}_{q,p}(g)$*

$$\begin{aligned} E_i(z) &= \mathrm{e}^{Q_i} z^{P_i} : \exp[\varphi_i(z(pq)^{1/2})] : \\ F_i(z) &= \mathrm{e}^{-Q_i} z^{-P_i} : \exp[-\psi_i(zq^{1/2})] : \\ H_i^+(z) &=: E_i(zp^{1/4}) F_i(zp^{-1/4}) : \\ H_i^-(z) &=: E_i(zp^{-1/4}) F_i(zp^{1/4}) : \end{aligned}$$

where : : means taking all subexpressions consisting of $a_i[n]$ with $n > 0$ and P_i to the right of expressions consisting of $a_i[n]$ with $n < 0$ and Q_i.

The proof for this proposition is also by straightforward but tedious calculations.

5. Concluding remarks

In this paper we have obtained the new elliptic current algebras $\mathcal{E}_{q,p}(\hat{g})$ and showed that these algebras have a structure of an infinite Hopf family of algebras. So far we have obtained two kinds of non-trivial infinite Hopf family of algebras: trigonometric (for the algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$) and elliptic (for the algebras $\mathcal{E}_{q,p}(\hat{g})$). It is thus an interesting question to ask whether there exists any rational algebras which have the same co-algebraic structure.

It is interesting that the co-multiplications appearing in such co-structures are all of Drinfeld-type, which closes over the currents themselves and does not require resolution of the inverse problem (Riemann problem) of the Ding–Frenkel homomorphism. We recall that two kinds of co-multiplications (and thus two kinds of Hopf algebra structures) are known for the standard q-affine algebras. The algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ and $\mathcal{E}_{q,p}(\hat{g})$ should be considered as some deformation of q-affine algebras and under such deformations the difference between the two Hopf algebra structures for q-affine algebras become clear: the standard Hopf structure for q-affine algebras is inherited into Yang–Baxter-type realizations for the algebras $\mathcal{A}_{q,p}(\hat{sl}_2)$ and $\mathcal{B}_{q,\lambda}(\hat{g})$ [12] and define the quasi-triangular quasi-Hopf structures in those algebras. Drinfeld-type Hopf structure is inherited into the current realizations for the algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ and $\mathcal{E}_{q,p}(\hat{g})$ and gives rise to the structure of an infinite Hopf family of algebras. The relation between the quasi-triangular quasi-Hopf structure and the infinite Hopf family of algebras is an interesting open problem to be answered in later studies.

We should emphasize that this work is only a preliminary study for the algebras $\mathcal{E}_{q,p}(\hat{g})$ themselves. Besides the definition and level 1 bosonic realization, we know very little about these algebras, especially their detailed representation theory, vertex operators, Yang–Baxter type realizations etc. The physical applications should also be considered.

Finally, the structures of an infinite Hopf family of algebras is still poorly understood. We do not know whether there exists a quantum double construction over the infinite Hopf family of algebras and, if not, what kind of new structure will take the place of the standard quantum doubles. Also, the classical counterpart of the infinite Hopf family of algebras is unknown and all these problems warrant further investigations.

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